

LOEWNER EQUATIONS ON COMPLETE HYPERBOLIC DOMAINS

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ABSTRACT. We prove that, on a complete hyperbolic domain $D \subset \mathbb{C}^q$, any Loewner PDE associated with a Herglotz vector field of the form $H(z, t) = \Lambda(z) + O(|z|^2)$, where the eigenvalues of Λ have strictly negative real part, admits a solution given by a family of univalent mappings $(f_t: D \rightarrow \mathbb{C}^q)$ which satisfies $\cup_{t \geq 0} f_t(D) = \mathbb{C}^q$. If no real resonance occurs among the eigenvalues of Λ , then the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin. We also give a generalization of Pommerenke's univalence criterion on complete hyperbolic domains.

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1. INTRODUCTION

We begin recalling the Loewner equations on the unit disc $\mathbb{D} \subset \mathbb{C}$. The Loewner PDE is the following:

$$\frac{\partial f_t(z)}{\partial t} = -\frac{\partial f_t(z)}{\partial z} H(z, t), \quad \text{a.e. } t \geq 0, z \in \mathbb{D}, \quad (1.1)$$

where $H(z, t) = zp(z, t)$ and $p(z, t): \mathbb{D} \times \mathbb{R}^+ \rightarrow \mathbb{C}$ is measurable in $t \geq 0$, holomorphic in $z \in \mathbb{D}$ and satisfies $\operatorname{Re} p(z, t) < 0$ and $p(0, t) = -1$ for all $t \geq 0$. The second equation is

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the Loewner ODE:

$$\begin{cases} \frac{\partial}{\partial t} \varphi_{s,t}(z) = H(\varphi_{s,t}(z), t), & \text{a.e. } t \in [s, \infty), z \in \mathbb{D}, \\ \varphi_{s,s}(z) = z, & s \geq 0, z \in \mathbb{D}. \end{cases} \quad (1.2)$$

Both equations were introduced by Loewner in 1923 [17] and used to prove the case $n = 3$ of the Bieberbach conjecture. Loewner theory was developed by Pommerenke [20] and Kufarev [16] as a powerful tool in geometric function theory. In fact it is one of the main ingredients of the proof of the Bieberbach conjecture given by de Branges [8] in 1985. Among the extensions of the theory we recall the celebrated theory of Schramm-Loewner evolution [23] introduced in 1999.

Loewner theory was extended to several complex variables by Duren, Graham, Hamada, G. Kohr, M. Kohr, Pfaltzgraff and others [9][12][19]. Recently Bracci, Contreras and Díaz-Madrigal [5][6] (see also [3]) proposed a generalization of the Loewner ODE which has its natural setting in complete hyperbolic manifolds. In the following we denote by D a complete hyperbolic (in the sense of Kobayashi) domain of \mathbb{C}^q . Recall that a holomorphic vector field $H: D \rightarrow \mathbb{C}^q$ is said an *infinitesimal generator* provided the Cauchy problem

$$\begin{cases} \dot{z}(t) = H(z(t)), \\ z(0) = z_0 \end{cases}$$

has a solution $z: [0, +\infty) \rightarrow D$ for all $z_0 \in D$.

A *Herglotz vector field* on a complete hyperbolic domain $D \subset \mathbb{C}^q$ is a non-autonomous holomorphic vector field $H(z, t): D \times \mathbb{R}^+ \rightarrow \mathbb{C}^q$ which is measurable in $t \geq 0$, which is an infinitesimal generator for a.e. $t \geq 0$ fixed, and such that for any compact set $K \subset D$ there exists a function $c_K \in L_{loc}^d(\mathbb{R}^+, \mathbb{R}^+)$, with $d \in [1, \infty]$, such that

$$|H(z, t)| \leq c_K(t), \quad z \in K, t \geq 0.$$

These vector fields are the natural generalizations of the function $H(z, t) = -zp(z, t)$ in (1.1). The Loewner ODE studied in [3][6]

$$\begin{cases} \frac{\partial}{\partial t} \varphi_{s,t}(z) = H(\varphi_{s,t}(z), t), & \text{a.e. } t \in [s, \infty), z \in D, \\ \varphi_{s,s}(z) = z, & s \geq 0, z \in D, \end{cases} \quad (1.3)$$

has a locally absolutely continuous (in the variable t) solution defined for all $0 \leq s \leq t$ given by a family $(\varphi_{s,t}: D \rightarrow D)$ of univalent mappings which is a \mathbb{R}^+ -evolution family, that is which satisfies $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t$ and $\varphi_{s,s}(z) = z$ for all $s \geq 0$.

Let $H(z, t)$ be a Herglotz vector field on D . In [4] we proved that a family of univalent mappings $(f_t: D \rightarrow \mathbb{C}^q)$ is locally absolutely continuous (in the variable t) and solves the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -df_t(z)H(z, t), \quad \text{a.e. } t \geq 0, z \in D, \quad (1.4)$$

if and only if it solves the functional equation

$$f_t \circ \varphi_{s,t}(z) = f_s(z), \quad 0 \leq s \leq t, \quad z \in D, \quad (1.5)$$

where $(\varphi_{s,t})$ is the solution of (1.3).

The solution $(f_t: D \rightarrow \mathbb{C}^q)$ satisfies $f_s(D) \subset f_t(D)$ for all $0 \leq s \leq t$. A family of univalent mappings with this property is called a \mathbb{R}^+ -Loewner chain. A \mathbb{R}^+ -evolution family $(\varphi_{s,t})$ and a \mathbb{R}^+ -Loewner chain (f_t) are associated if (1.5) holds.

We now introduce the special Herglotz vector fields that we are going to study in this paper. A Herglotz vector field $H(z, t)$ is of *dilation type* if

$$H(z, t) = \Lambda(z) + O(|z|^2), \quad t \geq 0,$$

where the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part, and the term $O(|z|^2)$ may depend on t .

We recall the following recent result by Graham, Hamada, G. Kohr and M. Kohr [12].

Theorem 1.1 ([12]). *Let $H(z, t) = \Lambda(z) + O(|z|^2)$ be a dilation Herglotz vector field on the unit ball $\mathbb{B} \subset \mathbb{C}^q$, and assume that*

$$2 \max\{\operatorname{Re} \langle \Lambda(z), z \rangle : |z| = 1\} < \min\{\operatorname{Re} \lambda : \lambda \in \operatorname{sp}(\Lambda)\}. \quad (1.6)$$

Then the Loewner PDE (1.4) admits a locally absolutely continuous univalent solution $(f_t: \mathbb{B} \rightarrow \mathbb{C}^q)$ such that $\cup_{t \geq 0} f_t(\mathbb{B}) = \mathbb{C}^q$. The family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin.

In [2] we introduced \mathbb{N} -evolution families and \mathbb{N} -Loewner chains, that is the discrete-time analogues of \mathbb{R}^+ -evolution families and \mathbb{R}^+ -Loewner chains. Solving equation (1.5) for discrete times we proved the following result.

Theorem 1.2 ([2]). *Let $H(z, t) = \Lambda(z) + O(|z|^2)$ be a dilation Herglotz vector field on the unit ball $\mathbb{B} \subset \mathbb{C}^q$. Assume that Λ is diagonal. Then the Loewner PDE (1.4) admits a locally absolutely continuous univalent solution $(f_t: \mathbb{B} \rightarrow \mathbb{C}^q)$ such that $\cup_{t \geq 0} f_t(\mathbb{B}) = \mathbb{C}^q$. The family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin if no real resonance of the form*

$$\operatorname{Re} \left(\sum_{j=1}^q k_j \alpha_j \right) = \operatorname{Re} \alpha_l, \quad k_j \geq 0, \quad \sum_{j=1}^q k_j \geq 2$$

occurs among the eigenvalues (α_j) of Λ .

The same result was obtained independently with different methods by Voda [24], assuming $\max\{\operatorname{Re} \langle \Lambda(z), z \rangle : |z| = 1\} < 0$ instead of assuming Λ diagonal.

Notice that condition (1.6) avoids real resonances. Recall that by [2, Counterexample 2] the Loewner PDE associated with the autonomous dilation Herglotz vector field on $\mathbb{B} \subset \mathbb{C}^2$

$$H(z, t) = (\alpha z_1, 2\alpha z_2 + c z_1^2),$$

where $|\alpha| < 1/2$ and $c \in \mathbb{C}^*$ is small enough, does not admit any solution $(f_t: \mathbb{B} \rightarrow \mathbb{C}^q)$ such that the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin. In this case a real resonance occurs.

In this paper we generalize Theorem 1.2 to any dilation Herglotz vector field on a complete hyperbolic domain $D \subset \mathbb{C}^q$. We should mention that, to our knowledge, this is the first existence result for the Loewner PDE (1.4) on such domains. We start by solving equation (1.5) for discrete times. Let $(\varphi_{n,m})$ be a \mathbb{N} -evolution family. We show that a family of tangent to identity univalent mappings $(h_n: D \rightarrow \mathbb{C}^q)$ which is uniformly bounded near the origin solves the *non-autonomous Schröder equation*

$$h_m \circ \varphi_{n,m} = e^{\Lambda(m-n)} \circ h_n. \quad (1.7)$$

if and only if $(\varphi_{n,m})$ is associated with the \mathbb{N} -Loewner chain $(f_n) \doteq (e^{-\Lambda n} \circ h_n)$.

Equation (1.7) shows a strong connection between Loewner theory and the theory of basins of attraction of discrete non-autonomous complex dynamical systems grown around Bedford's conjecture: see [1][11][14][18][25][22]. To solve equation (1.7) we use techniques from this theory, in particular from [18]. Indeed we need a non-autonomous version of the Poincaré-Dulac method, whose homological equation is replaced by a difference equation in the space \mathcal{H}_i of homogeneous polynomial mappings of degree i ,

$$H_{n+1} = e^\Lambda \circ H_n \circ e^{-\Lambda} + B_n, \quad (1.8)$$

where (H_n) is an unknown bounded sequence in \mathcal{H}_i and (B_n) is a bounded sequence in \mathcal{H}_i . In order to find a bounded solution of (1.8) we study the spectral and dynamical properties of the linear operator $H \mapsto e^\Lambda \circ H \circ e^{-\Lambda}$ acting on \mathcal{H}_i and we show that the obstruction to the existence of solutions is given by real resonances.

This method provides a family of univalent mappings $(f_n: r\mathbb{B} \subset D \rightarrow \mathbb{C}^q)_{n \in \mathbb{N}}$ satisfying (1.5) but defined only for integer times and in a little neighborhood of the origin. Then we extend this family to all $t \in \mathbb{R}^+$ and $z \in D$.

The main result of this paper is thus the following.

Theorem 1.3. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain and let $H(z, t) = \Lambda(z) + O(|z|^2)$ be a dilation Herglotz vector field on D . Then the Loewner PDE (1.4) admits a locally absolutely continuous univalent solution $(f_t: \mathbb{B} \rightarrow \mathbb{C}^q)$ such that $\cup_{t \geq 0} f_t(D) = \mathbb{C}^q$. The family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin if no real resonance occurs among the eigenvalues of Λ .*

We also generalize to complete hyperbolic domains the classical univalence criterion in the unit disk due to Pommerenke [20, Folgerung 6].

Theorem 1.4. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain and let $H(z, t) = \Lambda(z) + O(|z|^2)$ be a dilation Herglotz vector field on D . Let $(f_t: D \rightarrow \mathbb{C}^q)$ be a family of holomorphic mappings which solves the Loewner PDE (1.4) and assume that the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin is an univalent family. Then for all $t \geq 0$ the mapping f_t is univalent.*

2. LOCAL CONJUGACY

We start recalling some basic definitions.

Definition 2.1. Let \mathbb{T} be \mathbb{N} or \mathbb{R}^+ . Let D be a domain of \mathbb{C}^q . A \mathbb{T} -evolution family is a family of univalent mappings $(\varphi_{\alpha,\beta}: D \rightarrow D)_{\alpha \leq \beta \in \mathbb{T}}$ such that

- i) $\varphi_{\alpha,\alpha} = \text{id}$ for all $\alpha \geq 0$,
- ii) $\varphi_{\alpha,\beta} = \varphi_{\gamma,\beta} \circ \varphi_{\alpha,\gamma}$ for all $0 \leq \alpha \leq \gamma \leq \beta$.

A family of univalent mappings $(f_\alpha: D \rightarrow \mathbb{C}^q)$ is a \mathbb{T} -Loewner chain if $f_\alpha(D) \subset f_\beta(D)$ for all $0 \leq \alpha \leq \beta$.

Remark 2.2. Let $(f_\alpha: D \rightarrow \mathbb{C}^q)$ be a \mathbb{T} -Loewner chain. Then there exists a unique associated \mathbb{T} -evolution family $(\varphi_{\alpha,\beta} \doteq f_\beta^{-1} \circ f_\alpha)$.

One has the following uniqueness result for \mathbb{T} -Loewner chains.

Theorem 2.3 ([4]). *Let $(\varphi_{\alpha,\beta})$ be a \mathbb{T} -evolution family on D and let $(f_\alpha: D \rightarrow \mathbb{C}^q)$ be an associated \mathbb{T} -Loewner chain. If $(g_\alpha: D \rightarrow \mathbb{C}^q)$ is a subordination chain associated with $(\varphi_{\alpha,\beta})$ then there exists a holomorphic mapping $\Psi: \cup_{\alpha \in \mathbb{T}} f_\alpha(D) \rightarrow \mathbb{C}^q$ such that $(g_\alpha = \Psi \circ f_\alpha)$.*

In what follows we focus on special types of \mathbb{N} -evolution families and \mathbb{N} -Loewner chains.

Definition 2.4. We denote $\mathcal{L}(\mathbb{C}^q)$ and $\mathcal{A}(\mathbb{C}^q)$ the sets of \mathbb{C} -linear endomorphisms and \mathbb{C} -linear automorphisms of \mathbb{C}^q . Let $A \in \mathcal{L}(\mathbb{C}^q)$. The *spectrum* $\sigma(A)$ of A is the set of its eigenvalues. The *spectral radius* $\rho(A)$ is defined as $\max_{\lambda \in \sigma(A)} |\lambda|$.

Let D be a domain in \mathbb{C}^q containing 0. A \mathbb{N} -evolution family $(\varphi_{n,m})$ on D is a *dilation \mathbb{N} -evolution family* if for all $n \geq 0$,

$$\varphi_{n,n+1}(z) = A(z) + O(|z|^2), \quad (2.1)$$

with $A \in \mathcal{A}(\mathbb{C}^q)$ such that $\rho(A) < 1$. A \mathbb{N} -Loewner chain $(f_n: D \rightarrow \mathbb{C}^q)$ is a *locally bounded \mathbb{N} -Loewner chain* if for all $n \geq 0$,

$$f_n(z) = A^{-n}(z) + O(|z|^2),$$

where $A \in \mathcal{A}(\mathbb{C}^q)$ is such that $\rho(A) < 1$ and the family $(A^n \circ f_n)$ is uniformly bounded in a neighborhood of the origin.

On complete hyperbolic domains, the dynamics of dilation \mathbb{N} -evolution families is uniformly contractive, as the following lemma shows. A reference for complete hyperbolic manifolds and the Kobayshi distance is [15].

Lemma 2.5. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain and let $(\varphi_{n,m})$ be dilation \mathbb{N} -evolution family on D . Then the basin of attraction of the origin at time $n \geq 0$*

$$\mathfrak{A}(n) \doteq \{z \in D : \lim_{m \rightarrow \infty} \varphi_{n,m}(z) = 0\}$$

is the whole D , and for all $n \geq 0$ the convergence $\lim_{m \rightarrow \infty} \varphi_{n,m}(z) = 0$ is uniform on compact subsets.

Proof. Up to a linear change of coordinates, we may assume that $\max_{z \in \mathbb{C}^q} \frac{|A(z)|}{|z|} < 1$. Lemma A.2 yields then that there exists $\varepsilon > 0$ such that the Kobayashi ball $\Omega(0, \varepsilon)$ centered in the origin of radius ε is contained in the set $\bigcap_{m \geq 0} \mathfrak{A}(m)$. For all $n \geq 0$, the set $\mathfrak{A}(n)$ is an open subset of D . Indeed, if $z \in \mathfrak{A}(n)$, there exists $m > 0$ such that $\varphi_{n,m}(z) \in \Omega(0, \varepsilon/2)$. Since holomorphic mappings decrease the Kobayashi distance, one has

$$\varphi_{n,m}(\Omega(z, \varepsilon/2)) \subset \Omega(0, \varepsilon) \subset \mathfrak{A}(m),$$

thus $\Omega(z, \varepsilon/2) \subset \mathfrak{A}(n)$.

The set $\mathfrak{A}(n)$ is also a closed subset of D . Indeed let z be a point in the closure of $\mathfrak{A}(n)$. Then there exist a point $w \in \mathfrak{A}(n)$ such that $k_D(z, w) < \varepsilon/2$. Let $u > 0$ be such that $\varphi_{n,u}(w) \in \Omega(0, \varepsilon/2)$. Since holomorphic mappings decrease the Kobayashi distance one has

$$\varphi_{n,u}(z) \in \Omega(0, \varepsilon) \subset \mathfrak{A}(u),$$

thus $z \in \mathfrak{A}(n)$.

Since D is connected one has $\mathfrak{A}(n) = D$. The convergence is local uniform and hence uniform on compact subsets. □

A *triangular mapping* is a mapping $T: \mathbb{C}^q \rightarrow \mathbb{C}^q$ whose components $T^{(i)}(z)$ satisfy

$$T^{(1)}(z) = \lambda_1 z_1, \quad T^{(i)}(z) = \lambda_i z_i + t^{(i)}(z_1, z_2, \dots, z_{i-1}), \quad 2 \leq i \leq q,$$

where $\lambda_i \in \mathbb{C}$ and $t^{(i)}$ is a polynomial in $i - 1$ variables fixing the origin. Its *degree* is the maximum of the degree of its components. If $\lambda_i \neq 0$ for all $1 \leq i \leq q$, the mapping T is called a *triangular automorphism*. This is indeed an automorphism of \mathbb{C}^q , since we can iteratively write its inverse, which is still a triangular automorphism. Since the composition of two triangular automorphisms is still a triangular automorphism, they form a subgroup of $\text{aut}(\mathbb{C}^q)$. A *triangular dilation \mathbb{N} -evolution family* is a dilation \mathbb{N} -evolution family $(T_{n,m}, \mathbb{C}^q)$ such that each $T_{n,n+1}$, and hence every $T_{n,m}$, is a triangular automorphism of \mathbb{C}^q . A triangular dilation \mathbb{N} -evolution family $(T_{n,m})$ has *uniformly bounded coefficients* if the family $(T_{n,n+1})$ has uniformly bounded coefficients. A triangular dilation \mathbb{N} -evolution family $(T_{n,m})$ has *uniformly bounded degree* if the family $(T_{n,n+1})$ has uniformly bounded degree.

Definition 2.6. Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. A dilation \mathbb{N} -evolution family $(\varphi_{n,m}: D \rightarrow D)$ and a triangular dilation \mathbb{N} -evolution family $(T_{n,m})$ with uniformly bounded degree and uniformly bounded coefficients are *locally conjugate* if there exists, on a ball $r\mathbb{B} \subset D$ satisfying

$$\varphi_{n,m}(r\mathbb{B}) \subset (r\mathbb{B}), \quad 0 \leq n \leq m,$$

a uniformly bounded family of holomorphic mappings $(h_n: r\mathbb{B} \rightarrow \mathbb{C}^q)$ such that $h_n(z) = z + O(|z|^2)$ for all $n \geq 0$, and such that

$$h_m \circ \varphi_{n,m}(z) = T_{n,m} \circ h_n(z), \quad z \in r\mathbb{B}, \quad 0 \leq n \leq m. \quad (2.2)$$

Proposition 2.7. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Assume that a dilation \mathbb{N} -evolution family $(\varphi_{n,m}: D \rightarrow D)$ and a triangular dilation \mathbb{N} -evolution family $(T_{n,m})$ with uniformly bounded degree and uniformly bounded coefficients are locally conjugate by $(h_n: r\mathbb{B} \rightarrow \mathbb{C}^q)$. Then for each fixed $n \geq 0$ the sequence $(T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m})_{m \geq n}$ is eventually defined on each compact subset $K \subset D$, its limit*

$$h_n^e \doteq \lim_{m \rightarrow \infty} T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m}$$

exists uniformly on compacta on D , and satisfies $h_n^e|_{r\mathbb{B}} = h_n$. The family $(h_n^e: D \rightarrow \mathbb{C}^q)$ satisfies

$$h_m^e \circ \varphi_{n,m}(z) = T_{n,m} \circ h_n^e(z), \quad z \in D, \quad 0 \leq n \leq m.$$

Proof. Let $K \subset D$ be a compact subset. By Lemma 2.5, for all $n \geq 0$ there exists $u \geq n$ such that $\varphi_{n,u}(K) \subset r\mathbb{B}$. Then for $m \geq u$,

$$T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m}|_K = T_{n,u}^{-1} \circ (T_{u,m}^{-1} \circ h_m \circ \varphi_{u,m}) \circ \varphi_{n,u}|_K = T_{n,u}^{-1} \circ h_u \circ \varphi_{n,u}|_K$$

by (2.2), thus the sequence $(T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m})_{m \geq n}$ converges uniformly on compacta. By (2.2) we have,

$$T_{n,m}^{-1} \circ h_m \circ \varphi_{n,m}(z) = h_n(z), \quad z \in r\mathbb{B}, \quad n \leq m,$$

thus $h_n^e|_{r\mathbb{B}} = h_n$.

Finally

$$h_m^e \circ \varphi_{n,m} = \lim_{j \rightarrow \infty} T_{m,j}^{-1} \circ h_j \circ \varphi_{m,j} \circ \varphi_{n,m} = T_{n,m} \circ \lim_{j \rightarrow \infty} T_{n,j}^{-1} \circ h_j \circ \varphi_{n,j} = T_{n,m} \circ h_n^e.$$

□

Definition 2.8. We call the mappings h_n^e *intertwining mappings*. Notice that since $h_n^e|_{r\mathbb{B}} = h_n$, the family $(h_n^e: D \rightarrow \mathbb{C}^q)$ is uniformly bounded in a neighborhood of the origin. From now on we will denote h_n^e simply by h_n .

Proposition 2.9. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Assume that a dilation \mathbb{N} -evolution family $(\varphi_{n,m}: D \rightarrow D)$ and a triangular dilation \mathbb{N} -evolution family $(T_{n,m})$ with uniformly bounded degree and uniformly bounded coefficients are locally conjugate. Then each intertwining mapping $h_n: D \rightarrow \mathbb{C}^q$ is univalent.*

Proof. Assume that there exist $z \neq w$ in D and $n \geq 0$ such that $h_n(z) = h_n(w)$. Then by (2.2),

$$h_m(\varphi_{n,m}(z)) = h_m(\varphi_{n,m}(w)), \quad 0 \leq n \leq m. \quad (2.3)$$

By Lemma A.3 there exists a ball $s\mathbb{B}$ such that for all $m \geq 0$ the mapping $h_m|_{s\mathbb{B}}$ is univalent. By Lemma 2.5 there exists $m \geq n$ such that $\varphi_{n,m}(z) \cup \varphi_{n,m}(w) \subset s\mathbb{B}$. But

$\varphi_{n,m}(z) \neq \varphi_{n,m}(w)$ since $\varphi_{n,m}$ is a univalent mapping, hence (2.3) contradicts the univalence of $h_m|_{s\mathbb{B}}$. \square

3. NON-AUTONOMOUS POINCARÉ-DULAC METHOD

For a detailed exposition of the classical Poincaré-Dulac method, see [21, Appendix]. We will need the non-autonomous version of the Poincaré-Dulac method developed in [18] in the case of \mathbb{N} -evolution families of holomorphic automorphisms of \mathbb{C}^q . We will give alternative proofs of this method in Propositions 3.4 and 3.6 in which we show also that, in absence of real resonances (defined below), it is possible to find a local conjugacy between a dilation \mathbb{N} -evolution family and its linear part.

In what follows we identify a linear automorphism $A \in \mathcal{A}(\mathbb{C}^q)$ with its associated matrix with respect to the canonical basis.

Definition 3.1. A *real multiplicative resonance* for $A \in \mathcal{A}(\mathbb{C}^q)$ with eigenvalues λ_i is an identity

$$|\lambda_j| = |\lambda_1^{i_1} \dots \lambda_q^{i_q}|,$$

where $i_j \geq 0$, and $\sum_j i_j \geq 2$. If for every $1 \leq j \leq q$ we have $|\lambda_j| < 1$, real multiplicative resonances can occur only in a finite number. Moreover, if $0 < |\lambda_q| \leq \dots \leq |\lambda_1| < 1$, then

$$|\lambda_j| = |\lambda_1^{i_1} \dots \lambda_q^{i_q}| \Rightarrow i_j = i_{j+1} = \dots = i_q = 0. \quad (3.1)$$

Definition 3.2. An automorphism $A \in \mathcal{A}(\mathbb{C}^q)$ is in *optimal form* if

- i) A is in lower-triangular ε -Jordan normal form for some $\varepsilon > 0$, that is in lower triangular Jordan normal form with the underdiagonal multiplied by ε ,
- ii) if the diagonal of A is $(\lambda_1, \dots, \lambda_q)$ then $1 > |\lambda_1| \geq \dots \geq |\lambda_q| > 0$,
- iii) one has $\max_{z \in \mathbb{C}^q} \frac{|A(z)|}{|z|} < 1$.

Note that any linear automorphism can be put in optimal form by a linear change of coordinates.

Let $A \in \mathcal{A}(\mathbb{C}^q)$ be in optimal form. For $1 \leq j \leq q$ let $\pi_j: \mathbb{C}^q \rightarrow \mathbb{C}$ be the projection to the j -th coordinate. Let $i \geq 2$ and let \mathcal{H}_i be the vector space of all holomorphic maps $H: \mathbb{C}^q \rightarrow \mathbb{C}^q$ whose components $\pi_j \circ H$ are homogeneous polynomials of degree i . A basis for this vector space is easily described: let $1 \leq j \leq q$, let $I \in \mathbb{N}^q$ be a multi-index of absolute value $|I| = i$, and define X_I^j such that

$$\pi_l \circ X_I^j \doteq \delta_{l,j} z^I, \quad 1 \leq l \leq q.$$

The set $\mathfrak{B} \doteq \{X_I^j : 1 \leq j \leq q, |I| = q\}$ is a basis of \mathcal{H}_i . Next we define a splitting of \mathcal{H}_i by specifying a partition of the basis \mathfrak{B} .

We set $X_I^j \in \mathfrak{B}_r$ if $|\lambda_j \lambda^{-I}| = 1$. The *real resonant subspace* \mathcal{R}_i is the vector subspace spanned by the vectors in \mathfrak{B}_r .

We set $X_I^j \in \mathfrak{B}_s$ if $|\lambda_j \lambda^{-I}| < 1$. The *stable subspace* \mathcal{S}_i is the vector subspace spanned by the vectors in \mathfrak{B}_s .

We set $X_I^j \in \mathfrak{B}_u$ if $|\lambda_j \lambda^{-I}| > 1$. The *unstable subspace* \mathcal{U}_i is the vector subspace spanned by the vectors in \mathfrak{B}_u .

This defines the splitting $\mathcal{H}_i = \mathcal{R}_i \oplus \mathcal{S}_i \oplus \mathcal{U}_i$, with projections π_r , π_s , and π_u .

If $F \in \mathcal{L}(\mathbb{C}^q)$, then $H \mapsto H \circ F$ and $H \mapsto F \circ H$ are endomorphisms of \mathcal{H}_i . We define the linear operator $\Gamma: \mathcal{H}_i \rightarrow \mathcal{H}_i$ as $H \mapsto A \circ H \circ A^{-1}$.

The next lemma justifies the terms “stable” and “unstable”.

Lemma 3.3. *The stable subspace \mathcal{S}_i is Γ -totally invariant and $\rho(\Gamma|_{\mathcal{S}_i}) < 1$. Indeed*

$$\text{sp}(\Gamma|_{\mathcal{S}_i}) = \{\lambda_j \lambda^{-I} : X_I^j \in \mathfrak{B}_s\}.$$

The unstable subspace \mathcal{U}_i is Γ -totally invariant and $\rho(\Gamma^{-1}|_{\mathcal{U}_i}) < 1$. Indeed

$$\text{sp}(\Gamma|_{\mathcal{U}_i}) = \{\lambda_j \lambda^{-I} : X_I^j \in \mathfrak{B}_u\}.$$

Proof. Since the Γ -invariance is an straightforward calculation, we prove the statement concerning the spectrum of $\Gamma|_{\mathcal{S}_i}$. The automorphism A is conjugate to any automorphism obtained multiplying the underdiagonal by a positive constant. Thus there exists a continuous path $\gamma: [0, 1] \rightarrow \mathcal{A}(\mathbb{C}^q)$ such that $\gamma(0) = A$ and $\gamma(1) = (\lambda_1 z_1, \dots, \lambda_q z_q)$, with $\gamma(0)$ conjugated to $\gamma(t)$ for all $t \in [0, 1]$.

Let $M \in \mathcal{A}(\mathbb{C}^q)$. Define $\Xi(M) \in \mathcal{A}(\mathcal{S}_i)$ as $H \mapsto M \circ H \circ M^{-1}$. If $B = M \circ A \circ M^{-1}$, the linear operator $\Gamma|_{\mathcal{S}_i} = \Xi(A)$ is conjugate to the linear operator $\Xi(B)$. Indeed

$$B \circ H \circ B^{-1} = M \circ A \circ M^{-1} \circ H \circ M \circ A^{-1} \circ M^{-1},$$

thus $\Xi(B) = \Xi(M) \circ \Xi(A) \circ \Xi(M)^{-1}$.

We have $\lim_{t \rightarrow 1} \Xi(\gamma(t)) = \Xi(\lambda_1 z_1, \dots, \lambda_q z_q)$ and $\Gamma|_{\mathcal{S}_i} = \Xi(A) = \Xi(\gamma(0))$ is conjugate to $\Xi(\gamma(t))$ for all $t \in [0, 1]$. Thus

$$\text{sp}(\Gamma|_{\mathcal{S}_i}) = \text{sp}(\Xi(\lambda_1 z_1, \dots, \lambda_q z_q)).$$

It is easy to see that the linear operator $\Xi(\lambda_1 z_1, \dots, \lambda_q z_q)$ is diagonalizable and that the basis \mathfrak{B}_s is a basis of eigenvectors such that

$$[\Xi(\lambda_1 z_1, \dots, \lambda_q z_q)](X_I^j) = \lambda_j \lambda^{-I} X_I^j.$$

Thus

$$\text{sp}(\Gamma|_{\mathcal{S}_i}) = \text{sp}(\Xi(\lambda_1 z_1, \dots, \lambda_q z_q)) = \{\lambda_j \lambda^{-I} : X_I^j \in \mathfrak{B}_s\}.$$

The same argument works for the spectrum of $\Gamma|_{\mathcal{U}_i}$. □

Proposition 3.4. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Let $(\varphi_{n,m}: D \rightarrow D)$ be a dilation \mathbb{N} -evolution family such that $\varphi_{n,n+1}(z) = A(z) + O(|z|^2)$ with A in optimal form. Then for each $i \geq 2$ there exist*

- i) *a family (k_n^i) of polynomial maps $k_n(z) = z + O(|z|^2)$ with uniformly bounded degree and uniformly bounded coefficients, and*

ii) a triangular dilation evolution family $(T_{n,m}^i)$ with $T_{n,n+1}^i(z) = A(z) + O(|z|^2)$,

$$\deg T_{n,n+1}^i \leq i - 1,$$

and uniformly bounded coefficients such that for all $n \geq 0$,

$$k_{n+1}^i \circ \varphi_{n,n+1} - T_{n,n+1}^i \circ k_n^i = O(|z|^i). \quad (3.2)$$

If no multiplicative real resonance occurs among the eigenvalues of A , then the family $(T_{n,m}^i)$ is the linear family (A^{m-n}) .

Proof. For $i = 2$ set $T_{n+1,n}^2 = A$, $k_n^2 = \text{id}$, and we are done since A is a triangular mapping. Now assume that (3.2) holds for $i \geq 2$. We can rewrite (3.2) as

$$k_{n+1}^i \circ \varphi_{n,n+1} - T_{n,n+1}^i \circ k_n^i = P_{n,n+1} + O(|z|^{i+1}), \quad (3.3)$$

where $(P_{n,n+1})$ is a bounded sequence in \mathcal{H}_i . Define $R_{n,n+1} \doteq \pi_r(P_{n,n+1})$ which is in the real resonant subspace \mathcal{R}_i , and $N_{n,n+1} \doteq P_{n,n+1} - R_{n,n+1} \in \mathcal{S}_i \oplus \mathcal{U}_i$. Set

$$T_{n,n+1}^{i+1} \doteq T_{n,n+1}^i + R_{n,n+1},$$

which is still a triangular dilation \mathbb{N} -evolution family with uniformly bounded degree and uniformly bounded coefficients since $R_{n,n+1}$ is a triangular mapping thanks to (3.1), and set

$$k_n^{i+1} \doteq k_n^i + H_n \circ k_n^i,$$

where (H_n) is an unknown bounded sequence in \mathcal{H}_i .

$$\begin{aligned} k_{n+1}^{i+1} \circ \varphi_{n,n+1} - T_{n,n+1}^{i+1} \circ k_n^{i+1} &= \\ &= (k_{n+1}^i + H_{n+1} \circ k_{n+1}^i) \circ \varphi_{n,n+1} - (T_{n,n+1}^i + R_{n,n+1}) \circ (k_n^i + H_n \circ k_n^i) \\ &= P_{n,n+1} - R_{n,n+1} + H_{n+1} \circ A - A \circ H_n + O(|z|^{i+1}) \\ &= N_{n,n+1} + H_{n+1} \circ A - A \circ H_n + O(|z|^{i+1}). \end{aligned}$$

Thus to end the proof we need to prove the existence of a bounded sequence (H_n) of elements of \mathcal{H}_i which satisfies

$$N_{n,n+1} = A \circ H_n - H_{n+1} \circ A, \quad (3.4)$$

that is a bounded solution (H_n) of the *homological difference equation*

$$H_{n+1} = A \circ H_n \circ A^{-1} - N_{n,n+1} \circ A^{-1}.$$

Define $B_n \doteq -N_{n,n+1} \circ A^{-1}$. In the proof of Lemma 3.3 we proved that \mathcal{S}_i and \mathcal{U}_i are invariant by the linear operator $H \mapsto H \circ A^{-1}$, thus $B_n \in \mathcal{S}_i \oplus \mathcal{U}_i$. Define $B_n^s \doteq \pi_s(B_n)$, $B_n^u \doteq \pi_u(B_n)$. If $n \geq 1$ it is easy to prove by induction that

$$H_n = \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}) = \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^{n-1-j}(B_j). \quad (3.5)$$

We have

$$\begin{aligned} H_n &= \Gamma^n(H_0) + \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^s) + \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^u) \\ &= \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^s) + \Gamma^{n-1}(\Gamma(H_0) + \sum_{j=0}^{n-1} \Gamma^{-j}(B_j^u)). \end{aligned}$$

Recall that if V is a complex vector space, and $L \in \mathcal{L}(V)$, then the spectral radius of L satisfies $\rho(L) = \inf_{\|\cdot\| \in \mathcal{I}} \{\|L\|\}$, where \mathcal{I} is the set of all operator norms induced by a norm on V . Hence by Lemma 3.3 there exist a norm $\|\cdot\|_s$ on \mathcal{S}_i and a norm $\|\cdot\|_u$ on \mathcal{U}_i such that $\|\Gamma|_{\mathcal{S}_i}\|_s < 1$, $\|\Gamma^{-1}|_{\mathcal{U}_i}\|_u < 1$. Define a norm on $\mathcal{S}_i \oplus \mathcal{U}_i$ by

$$\|H\| \doteq \|\pi_s(H)\|_s + \|\pi_u(H)\|_u.$$

Since (B_n^s) is bounded there exists $C > 0$ such that

$$\left\| \sum_{j=0}^{n-1} \Gamma^j(B_{n-1-j}^s) \right\| \leq \sum_{j=0}^{\infty} \|\Gamma^j(B_{n-1-j}^s)\|_s \leq C, \quad n \geq 0.$$

Since (B_n^u) is bounded, $\sum_{j=0}^{\infty} \|\Gamma^{-j}(B_j^u)\|_u < +\infty$, thus we can define

$$H_0 \doteq -\Gamma^{-1} \left(\sum_{j=0}^{\infty} \Gamma^{-j}(B_j^u) \right) \in \mathcal{U}_i.$$

With this definition,

$$\|H_n\| \leq C + \|\Gamma^{n-1} \left(\sum_{j=n}^{\infty} \Gamma^{-j}(B_j^u) \right)\|_u = C + \left\| \sum_{j=1}^{\infty} \Gamma^{-j}(B_{n-1+j}^u) \right\|_u,$$

and since

$$\left\| \sum_{j=1}^{\infty} \Gamma^{-j}(B_{n-1+j}^u) \right\|_u \leq \sum_{j=1}^{\infty} \|\Gamma^{-j}(B_{n-1+j}^u)\|_u \leq C',$$

we have $\|H_n\| \leq C + C'$.

If no multiplicative real resonance occurs among the eigenvalues of A , then

$$\mathcal{R}_i = \emptyset, \quad \text{for all } i \geq 2,$$

and thus $T_{n,m}^{i+1} = T_{n,m}^i$ for all $i \geq 2$, which gives

$$T_{n,m}^i = T_{n,m}^2 = A^{m-n}, \quad \text{for all } i \geq 2.$$

□

Remark 3.5. Let $p \geq 0$ be the smallest integer such that $|\lambda_1^p| < |\lambda_q|$. Then if $i \geq p$ we have $\pi_r(P_{n,n+1}) = 0$ in \mathcal{H}_i . Hence $T_{n,n+1}^i = T_{n,n+1}^p$ for any $i \geq p$.

Proposition 3.6. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Let $(\varphi_{n,m} : D \rightarrow D)$ be a dilation \mathbb{N} -evolution family such that $\varphi_{n,n+1}(z) = A(z) + O(|z|^2)$ with A in optimal form. Then there exists a triangular dilation \mathbb{N} -evolution family $(T_{n,m})$ with bounded degree and bounded coefficients locally conjugate to $(\varphi_{n,m})$. If no multiplicative real resonance occurs among the eigenvalues of A , then $(\varphi_{n,m})$ is locally conjugate to its linear part (A^{m-n}) .*

Proof. Let α be such that $\max_{z \in \mathbb{C}^q} \frac{|A(z)|}{|z|} < \alpha < 1$. Let $(T_{n,m}^i)$ and (k_n^i) be the families given by Proposition 3.4. Let $p \geq 0$ be as in previous remark. Define $(T_{n,m}) \doteq (T_{n,m}^p)$. Let $\beta > 0$ be the constant given by Lemma A.4 for $(T_{n,m})$. Let $\ell \geq 0$ be an integer such that $\alpha^\ell < 1/\beta$, and define $(k_n) \doteq (k_n^\ell)$. By Proposition 3.4,

$$k_{n+1} \circ \varphi_{n,n+1} - T_{n,n+1} \circ k_n = O(|z|^\ell),$$

thus

$$T_{n,n+1}^{-1} \circ k_{n+1} \circ \varphi_{n,n+1} - k_n = O(|z|^\ell).$$

By Lemma A.2 there exists $r > 0$ (we can assume $0 < r < 1/2$) such that on $r\mathbb{B}$ we have $|\varphi_{n,n+1}(z)| \leq \alpha|z|$ and $|T_{n,n+1}(z)| \leq \alpha|z|$ for all $n \geq 0$. Thus for $\zeta \in r\mathbb{B}$ we have

$$|\varphi_{0,m}(\zeta)| < r\alpha^m.$$

Thanks to Lemma A.1 there exists $C > 0$ such that on $r\mathbb{B}$,

$$|T_{m,m+1}^{-1} \circ k_{m+1} \circ \varphi_{m,m+1}(\zeta) - k_m(\zeta)| \leq C|\zeta|^\ell, \quad m \geq 0.$$

Hence

$$|T_{m,m+1}^{-1} \circ k_{m+1} \circ \varphi_{0,m+1}(\zeta) - k_m \circ \varphi_{0,m}(\zeta)| \leq C|\varphi_{0,m}(\zeta)|^\ell \leq Cr^\ell \alpha^{\ell m}.$$

Let Δ be the unit polydisc. There exists $s\mathbb{B} \subset r\mathbb{B}$ such that

$$T_{m,m+1}^{-1} \circ k_{m+1} \circ \varphi_{0,m+1}(s\mathbb{B}) \subset \frac{1}{2}\Delta$$

and

$$k_m \circ \varphi_{0,m}(s\mathbb{B}) \subset \frac{1}{2}\Delta.$$

Indeed the families (k_m) and $(T_{m,m+1}^{-1})$ are uniformly bounded on $r\mathbb{B}$ and thus equicontinuous in 0.

Hence Lemma A.4 applies to get on $s\mathbb{B}$,

$$|T_{0,m+1}^{-1} \circ k_{m+1} \circ \varphi_{0,m+1}(\zeta) - T_{0,m}^{-1} \circ k_m \circ \varphi_{0,m}(\zeta)| \leq Cr^\ell (\beta\alpha^\ell)^m.$$

Likewise it is easy to see that for all $m \geq n \geq 0$,

$$|T_{n,m+1}^{-1} \circ k_{m+1} \circ \varphi_{n,m+1}(\zeta) - T_{n,m}^{-1} \circ k_m \circ \varphi_{n,m}(\zeta)| \leq Cr^\ell (\beta\alpha^\ell)^{m-n}.$$

Since $\alpha^\ell < 1/\beta$ for all $n \geq 0$ there exists a holomorphic mapping h_n on $s\mathbb{B}$ such that

$$h_n = \lim_{m \rightarrow \infty} T_{n,m}^{-1} \circ k_m \circ \varphi_{n,m}$$

uniformly on compacta. Each h_n is bounded by $|k_n| + \sum_{j=0}^{\infty} Cr^{\ell}(\beta\alpha^{\ell})^j$, hence they are uniformly bounded. Moreover

$$h_m \circ \varphi_{n,m} = \lim_{j \rightarrow \infty} T_{m,j}^{-1} \circ k_j \circ \varphi_{m,j} \circ \varphi_{n,m} = \lim_{j \rightarrow \infty} T_{n,m} \circ T_{j,n} \circ k_j \circ \varphi_{n,j} = T_{n,m} \circ h_n.$$

If no multiplicative real resonance occurs among the eigenvalues of A , then $(T_{n,m}) = (A^{m-n})$. \square

We can now prove an existence result for \mathbb{N} -Loewner chains.

Proposition 3.7. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Let $(\varphi_{n,m}: D \rightarrow D)$ be a dilation \mathbb{N} -evolution family, $\varphi_{n,n+1}(z) = A(z) + O(|z|^2)$. Then there exists a \mathbb{N} -Loewner chain $(f_n: D \rightarrow \mathbb{C}^q)$ with $\cup_{n \geq 0} f_n(D) = \mathbb{C}^q$ associated with $(\varphi_{n,m})$, which is locally bounded if no multiplicative real resonance occurs among the eigenvalues of A .*

Proof. Up to a linear change of coordinates, we can assume that A is in optimal form. By Proposition 3.6 there exists a triangular dilation \mathbb{N} -evolution family $(T_{n,m})$ with bounded degree and bounded coefficients locally conjugate to $(\varphi_{n,m})$. Let (h_n) be the family of intertwining mappings given by Proposition 2.7. Then $(T_{0,n}^{-1} \circ h_n)$ is a \mathbb{N} -Loewner chain (f_n) associated with $(\varphi_{n,m})$. Indeed

$$T_{0,m}^{-1} \circ h_m \circ \varphi_{n,m}(z) = T_{0,m}^{-1} \circ T_{n,m} \circ h_n(z) = T_{0,n}^{-1} \circ h_n(z), \quad z \in D, 0 \leq n \leq m.$$

By Lemma A.3 there exists a ball $s\mathbb{B} \subset \bigcap_{n \geq 0} h_n(D)$. Hence

$$\bigcup_{n \geq 0} T_{0,n}^{-1}(h_n(D)) \supset \bigcup_{n \geq 0} T_{0,n}^{-1}(s\mathbb{B}) = \mathbb{C}^q$$

by Lemma A.4.

If no multiplicative real resonance occurs among the eigenvalues of A , then $(T_{n,m}) = (A^{m-n})$, and the chain $(A^{-n} \circ h_n)$ is locally bounded. \square

Now we can go back to the continuous-time setting.

Definition 3.8. A \mathbb{R}^+ -evolution family $(\varphi_{s,t})$ on D is a *dilation \mathbb{R}^+ -evolution family* if for all $0 \leq s \leq t$,

$$\varphi_{s,t}(z) = e^{\Lambda(t-s)}(z) + O(|z|^2), \quad (3.6)$$

where the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part.

A \mathbb{R}^+ -Loewner chain $(f_t: D \rightarrow \mathbb{C}^q)$ is a *locally bounded \mathbb{R}^+ -Loewner chain* if for all $t \geq 0$,

$$f_t(z) = e^{-\Lambda t}(z) + O(|z|^2),$$

where the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part and the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin.

If we restrict time to integer values in a dilation \mathbb{R}^+ -evolution family $(\varphi_{s,t})$ we obtain its *discretized* dilation \mathbb{N} -evolution family $(\varphi_{n,m})$. We have

$$\varphi_{n,n+1}(z) = e^{\Lambda}(z) + O(|z|^2).$$

An *additive real resonance* is an identity

$$\operatorname{Re} \left(\sum_{j=1}^N k_j \alpha_j \right) = \operatorname{Re} \alpha_l,$$

where $k_j \geq 0$ and $\sum_j k_j \geq 2$. Recall that α is an eigenvalue of Λ with algebraic multiplicity m if and only if e^α is an eigenvalue of e^Λ with algebraic multiplicity m . Hence additive real resonances of Λ correspond to multiplicative real resonances of e^Λ .

Lemma 3.9. *Let D be a complete hyperbolic domain. Let $(\varphi_{s,t}: D \rightarrow D)$ be a dilation \mathbb{R}^+ -evolution family, and let $(\varphi_{n,m}: D \rightarrow D)$ be its discretized evolution family. Assume there exists a \mathbb{N} -Loewner chain (f_n) associated with $(\varphi_{n,m})$. Then we can extend it in a unique way to a \mathbb{R}^+ -Loewner chain associated with $(\varphi_{s,t})$. If (f_n) is a locally bounded \mathbb{N} -Loewner chain, then also (f_s) is locally bounded.*

Proof. For all $s \in \mathbb{R}^+$ define $f_s = f_j \circ \varphi_{s,j}$, where j is an integer such that $s \leq j$. The family (f_s) is a \mathbb{R}^+ -Loewner chain associated with $(\varphi_{s,t})$ (cf. [2, Lemma 8.5]). If (f_n) is a locally bounded \mathbb{N} -Loewner chain, then there exists $r > 0$ such that the family $(e^{\Lambda n} \circ f_n)$ is uniformly bounded on the Kobayashi ball $\Omega(0, r)$ centered in the origin of radius $r > 0$. For each $s \geq 0$ define m_s as the smallest integer greater than s . One has

$$e^{\Lambda s} \circ f_s = e^{\Lambda s} \circ f_{m_s} \circ \varphi_{s,m_s} = e^{\Lambda(s-m_s)} \circ e^{\Lambda m_s} \circ f_{m_s} \circ \varphi_{s,m_s},$$

and since $\varphi_{s,m_s}(\Omega(0, r)) \subset \Omega(0, r)$ and $m_s - s \leq 1$, the family $(e^{\Lambda s} \circ f_s)$ is uniformly bounded on $\Omega(0, r)$. \square

Proposition 3.10. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. Let $(\varphi_{s,t}: D \rightarrow D)$ be a dilation \mathbb{R}^+ -evolution family, $\varphi_{s,t}(z) = e^{\Lambda(t-s)}(z) = O(|z|^2)$. Then there exists a \mathbb{R}^+ -Loewner chain $(f_s: D \rightarrow \mathbb{C}^q)$ with $\cup_{t \geq 0} f_t(D) = \mathbb{C}^q$ associated with $(\varphi_{s,t})$, which is locally bounded if no additive real resonance occurs among the eigenvalues of Λ .*

Proof. Let $(\varphi_{n,m}: D \rightarrow D)$ be the discretized evolution family of $(\varphi_{s,t}: D \rightarrow D)$. Since no additive real resonance occurs in Λ , no multiplicative real resonance occurs in $A = e^\Lambda$. The result follows from Proposition 3.7 and Lemma 3.9. \square

Remark 3.11. If no additive real resonance occurs among the eigenvalues of Λ then by the proof of Proposition 3.10 there exists a family (h_n) of tangent to identity polynomial mappings of uniformly bounded degree and uniformly bounded coefficients such that

$$f_s = \lim_{m \rightarrow \infty} e^{-\Lambda m} \circ h_m \circ \varphi_{s,m}.$$

4. THE LOEWNER PDE

Definition 4.1. Let D be a domain in \mathbb{C}^q containing 0 and let $d \in [1, +\infty]$. A *dilation Herglotz vector field of order $d \geq 1$* on D is a mapping

$$G: D \times \mathbb{R}^+ \rightarrow \mathbb{C}^q$$

satisfying

- a) for all $z \in D$ the map $t \mapsto H(z, t)$ is measurable,
- b) for a.e. $t \geq 0$ the map $z \mapsto H(z, t)$ is an infinitesimal generator on D of the form

$$H(z, t) = \Lambda(z) + O(|z|^2)$$

where the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part.

- c) for any compact set $K \subset D$ and there exists a function $c_K \in L_{loc}^d(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$|H(z, t)| \leq c_K(t), \quad z \in K, t \geq 0.$$

The partial differential equation

$$\frac{\partial f_t(z)}{\partial t} = -df_t(z)H(z, t) \quad \text{a.e. } t \geq 0, z \in D, \quad (4.1)$$

where $H(z, t)$ is a dilation Herglotz vector field, is called the *Loewner PDE*.

The following is our main result.

Theorem 4.2. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain and let $H(z, t) = \Lambda(z) + O(|z|^2)$ be a dilation Herglotz vector field on D of order $d \geq 1$. Then the Loewner PDE (4.1) admits a solution given by a family of univalent mappings $(f_t: D \rightarrow \mathbb{C}^q)$, such that $\cup_{t \geq 0} f_t(D) = \mathbb{C}^q$, and which is locally absolutely continuous of order d in the following sense: for any compact set $K \subset D$ there exists a function $k_K \in L_{loc}^d(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$|f_s(z) - f_t(z)| \leq \int_s^t k_K(\xi) d\xi, \quad z \in K, 0 \leq s \leq t. \quad (4.2)$$

If no additive real resonance occurs among the eigenvalues of Λ then the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin. Any locally absolutely continuous solution given by a family of holomorphic mappings $(g_t: D \rightarrow \mathbb{C}^q)$ is of the form $(\Psi \circ f_t)$, where $\Psi: \mathbb{C}^q \rightarrow \mathbb{C}^q$ is holomorphic.

Proof. Since by assumption D is complete hyperbolic, [3] yields that the solution of the Loewner ODE

$$\begin{cases} \frac{\partial}{\partial t} \varphi_{s,t}(z) = H(\varphi_{s,t}(z), t), & \text{a.e. } t \in [s, \infty), \\ \varphi_{s,s}(z) = z, & s \geq 0. \end{cases} \quad (4.3)$$

is a \mathbb{R}^+ -evolution family $(\varphi_{s,t}(z) = e^{\Lambda(t-s)}(z) + O(|z|^2))$ which is locally absolutely continuous of order d in the following sense: for any compact set $K \subset D$ there exists a function $C_K \in L_{loc}^d(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq \int_u^t C_K(\xi) d\xi, \quad z \in K, 0 \leq s \leq u \leq t. \quad (4.4)$$

By Proposition 3.10 there exists a \mathbb{R}^+ -Loewner chain $(f_t: D \rightarrow \mathbb{C}^q)$ with $\cup_{t \geq 0} f_t(D) = \mathbb{C}^q$ associated with $(\varphi_{s,t})$. By [4, Theorem 4.10] the chain (f_t) is of locally absolutely

continuous of order d , and by [4, Theorem 5.2] it solves the Loewner PDE (4.1). If no additive real resonance occurs among the eigenvalues of Λ then by Proposition 3.10 the chain (f_t) is locally bounded. Any locally continuous solution of (4.1) is by [4, Theorem 5.2] a \mathbb{R}^+ -Loewner chain associated with $(\varphi_{s,t})$, thus by Theorem 2.3 it is of the form $(\Psi \circ f_t)$, where $\Psi: \mathbb{C}^q \rightarrow \mathbb{C}^q$ is holomorphic. \square

5. AN UNIVALENCE CRITERION

In this section we generalize a classical univalence criterion in the unit disk due to Pommerenke. We first need to generalize the notion of locally bounded \mathbb{R}^+ -Loewner chains in order to include families of non-necessarily univalent mappings.

Definition 5.1. Let D be a domain in \mathbb{C}^q containing 0. A family $(f_t: D \rightarrow \mathbb{C}^q)$ is a *locally bounded \mathbb{R}^+ -subordination chain* if

- i) for all $0 \leq s \leq t$ there exists a holomorphic mapping $\varphi_{s,t} \in \text{Hol}(D, D)$ fixing 0 and satisfying $f_s = f_t \circ \varphi_{s,t}$, called *transition mapping*,
- ii) $f_t(z) = e^{-\Lambda t}(z) + O(|z|^2)$ for all $t \geq 0$, the eigenvalues of $\Lambda \in \mathcal{L}(\mathbb{C}^q)$ have strictly negative real part, and the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin.

Now we can state Pommerenke's criterion.

Theorem 5.2 ([20, Folgerung 6]). *If $(f_t: \mathbb{D} \rightarrow \mathbb{C})$ is a locally bounded \mathbb{R}^+ -subordination chain, then for all $t \geq 0$ the mapping f_t is univalent.*

This criterion has been generalized to the unit ball $\mathbb{B} \subset \mathbb{C}^q$, with different hypotheses, by Pfaltzgraft [19, Theorem 2.3], and by Graham and Kohr [13, Theorem 8.1.6]. To generalize Pommerenke's criterion we do not assume the subordination chain to solve a Loewner PDE: we only assume continuity. A locally bounded \mathbb{R}^+ -subordination chain $(f_t: D \rightarrow \mathbb{C}^q)$ is *continuous* if the mapping $t \mapsto f_t$ is continuous with respect to the topology of uniform convergence on compacta on $\text{Hol}(D, \mathbb{C}^q)$.

Proposition 5.3. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain. If $(f_t: D \rightarrow \mathbb{C}^q)$ is a continuous locally bounded \mathbb{R}^+ -subordination chain, then for all $t \geq 0$ the mapping f_t is univalent.*

Proof. We have to show that f_t is univalent for all $t \geq 0$. The identity principle yields that for all $0 \leq s \leq t$ there exists a unique transition mapping $\varphi_{s,t} \in \text{Hol}(D, D)$ satisfying $f_s = f_t \circ \varphi_{s,t}$ and fixing 0, since f_t is locally invertible at 0. Thus the family $(\varphi_{s,t})_{0 \leq s \leq t}$ satisfies $\varphi_{s,s} = \text{id}$ for all $s \geq 0$. Moreover one has

$$\varphi_{u,t} \circ \varphi_{s,u} = \varphi_{s,t}, \quad 0 \leq s \leq u \leq t.$$

Indeed for all $0 \leq s \leq u \leq t$,

$$f_t \circ \varphi_{u,t} \circ \varphi_{s,u} = f_u \circ \varphi_{s,u} = f_s = f_t \circ \varphi_{s,t},$$

and the assertion follows since the transition mapping is unique. Moreover by the chain rule $df_s(0) = df_t(0) \circ d\varphi_{s,t}(0)$, hence $d\varphi_{s,t}(0) = e^{\Lambda(t-s)}$.

We claim that $\lim_{t \rightarrow s+} \varphi_{s,t} = \text{id}$ for all $s \geq 0$. Indeed since $(\varphi_{s,t}: D \rightarrow D)_{0 \leq s \leq t}$ is a normal family and $\varphi_{s,t}(0) = 0$ for all $0 \leq s \leq t$, any sequence (φ_{s,t_n}) with $t_n \rightarrow s$ admits a subsequence $(\varphi_{s,t_{n_k}})$ converging on compacta to a mapping $\varphi \in \text{Hol}(D, D)$, and by $f_s = f_{t_{n_k}} \circ \varphi_{s,t_{n_k}}$ we obtain $f_s = f_s \circ \varphi$, thus $\varphi = \text{id}$. This proves that $\lim_{t \rightarrow s+} \varphi_{s,t} = \text{id}$. In the same way, $\lim_{s \rightarrow t-} \varphi_{s,t} = \text{id}$.

This implies that $\varphi_{s,t}$ is univalent for all $0 \leq s \leq t$. Indeed suppose there exists $0 < s < t$ and $z \neq w$ contained in a Kobayashi ball $\Omega(0, \ell)$ such that $\varphi_{s,t}(z) = \varphi_{s,t}(w)$. Set $r \doteq \inf\{u \in [s, t] : \varphi_{s,u}(z) = \varphi_{s,u}(w)\}$. Since $\lim_{u \rightarrow s+} \varphi_{s,u} = \text{id}$ uniformly on compacta, we have $r > s$. If $u \in (s, r)$,

$$\varphi_{u,r}(\varphi_{s,u}(z)) = \varphi_{u,r}(\varphi_{s,u}(w)),$$

and since $\varphi_{s,u}(z) \neq \varphi_{s,u}(w)$, the mapping $\varphi_{u,r}$ is not univalent on

$$\bigcup_{u \in (s, r)} \varphi_{s,u}(z) \cup \varphi_{s,u}(w) \subset \Omega(0, \ell).$$

Since D is complete hyperbolic, by [15, Proposition 1.1.9] one has $\Omega(0, \ell) \subset\subset D$. But $\lim_{u \rightarrow r-} \varphi_{u,r} = \text{id}$ uniformly on compacta which is a contradiction since the identity mapping is univalent.

Define $h_s \doteq e^{\Lambda s} \circ f_s$, for all $s \geq 0$. By hypothesis the family $(h_s: D \rightarrow \mathbb{C}^q)$ is uniformly bounded in a neighborhood of the origin. From $f_t \circ \varphi_{s,t} = f_s$ we obtain

$$h_t \circ \varphi_{s,t} = e^{\Lambda(t-s)} \circ h_s.$$

Fix $s \geq 0$. The dilation N-evolution family $(\varphi_{s+n, s+m})_{0 \leq n \leq m}$ is locally conjugate to $(e^{\Lambda(m-n+s)})_{0 \leq n \leq m}$ by means of the intertwining mappings $(h_{s+n})_{n \geq 0}$. By Proposition 2.9 the mapping h_s is univalent, thus $f_s = e^{-\Lambda s} \circ h_s$ is univalent. \square

As a corollary one easily obtains the following.

Corollary 5.4. *Let $D \subset \mathbb{C}^q$ be a complete hyperbolic domain and let $H(z, t) = \Lambda(z) + O(|z|^2)$ be a dilation Herglotz vector field on D . Let $(f_t: D \rightarrow \mathbb{C}^q)$ be a locally absolutely continuous family of holomorphic mappings which solves the Loewner PDE (4.1) and assume that the family $(e^{\Lambda t} \circ f_t)$ is uniformly bounded in a neighborhood of the origin. Then for all $t \geq 0$ the mapping f_t is univalent.*

APPENDIX A. AUSILIARY LEMMAS

For the convenience of the reader, we recall here some auxiliary Lemmas, in the form used in the proofs.

Lemma A.1. [2, Lemma 2.2] *Let $A \in \mathcal{L}(\mathbb{C}^q)$. Let \mathcal{F} be a family of holomorphic mappings $(f: r\mathbb{B} \rightarrow \mathbb{C}^q)$, bounded by $M > 0$, and let $k \geq 2$ such that $f(z) - A(z) = O(|z|^k)$ for all $f \in \mathcal{F}$. Then there exists $C_k > 0$ such that $|f(z) - A(z)| \leq C_k |z|^k$ for all $z \in r\mathbb{B}$.*

Lemma A.2. [2, Lemma 2.3] *Let $A \in \mathcal{L}(\mathbb{C}^q)$, and let D be a domain containing 0. Let \mathcal{F} be a family of holomorphic mappings $(f: D \rightarrow \mathbb{C}^q)$, bounded by $M > 0$, and satisfying $f(z) = A(z) + O(|z|^2)$. Let $\alpha > 0$ be such that $\max_{z \in \mathbb{C}^q} \frac{|A(z)|}{|z|} < \alpha$. Then there exists $s > 0$ such that if $f \in \mathcal{F}$ then $|f(z)| \leq \alpha|z|$ for all $|z| \leq s$.*

Lemma A.3. [2, Lemma 2.5] *Let $A \in \mathcal{A}(\mathbb{C}^q)$, and let D be a domain containing 0. Let \mathcal{F} be a family of holomorphic mappings $(f: D \rightarrow \mathbb{C}^q)$, bounded by $M > 0$, and satisfying $f(z) = A(z) + O(|z|^2)$. There exist $r > 0$ and $s > 0$ such that if $f \in \mathcal{F}$ then f is univalent on $r\mathbb{B}$, and $s\mathbb{B} \subset f(r\mathbb{B})$.*

The following Lemma is stated in [18, Lemma 11] as a simple generalization of [21, Lemma 1, Appendix]. A proof can be found in [2, Corollary 4.4, Lemma 4.5].

Lemma A.4. [2, Corollary 4.4] *Let $\Delta \subset \mathbb{C}^q$ be the unit polydisc.*

Let $(T_{n,m})$ be a triangular dilation \mathbb{N} -evolution family of uniformly bounded degree and uniformly bounded coefficients. Then

a) *there exists $\beta \geq 0$ such that for all $k \geq 0$,*

$$|T_{0,k}^{-1}(z) - T_{0,k}^{-1}(z')| \leq \beta^k |z - z'|, \quad z, z' \in \frac{1}{2}\Delta.$$

b) *$T_{0,n}(z) \rightarrow 0$ uniformly on compacta and for each neighborhood V of 0 we have $\bigcup_{n=1}^{\infty} T_{0,n}^{-1}(V) = \mathbb{C}^q$.*

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